Statistical properties of time-reversible triangular maps of the square

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Abstract. Time reversal symmetric triangular maps of the unit square are introduced with the property that the time evolution of one of their two variables is determined by a piecewise expanding map of the unit interval. We study their statistical properties and establish the conditions under which their equilibrium measures have a product structure, *i.e.* factorises in a symmetric form. When these conditions are not verified, the equilibrium measure does not have a product form and therefore provides additional information on the statistical properties of theses maps. This is the case of anti-symmetric cusp maps, which have an intermittent fixed point and yet have uniform invariant measures on the unit interval. We construct the invariant density of the corresponding two-dimensional triangular map and prove that it exhibits a singularity at the intermittent fixed point.

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It has long been realized that chaos is a ubiquitous property of non-linear mechanical systems. The study of the dynamical properties of higher dimensional systems such as those encountered in the framework of statistical physics is however difficult from a theoretical stand-point, and it is therefore rather naturally that the theory of chaotic dynamical systems and their statistical properties was developed in the framework of low-dimensional systems. The ergodic and dynamical properties of one-dimensional piecewise expanding maps of the interval have been extensively studied in this regard [1] and, together with Anosov diffeomorphisms, were central to the derivation of some key results, in particular relating to the Sinaï-Ruelle-Bowen theory of natural invariant measures. See references [2, 3].

A central problem in the study of statistical properties of one-dimensional piecewise expanding maps is the identification of a natural invariant measure. A standard procedure, which applies to maps with the Markov property, is to establish a correspondence between the iterations of such maps and shifts on semi-infinite sequences of spin variables. This way the study of the invariant state of the map reduces to that of the corresponding lattice gas. One subsequently constructs a time translation invariant state by extending the semi-infinite spin system to one which is infinite in both directions [4]. As far as the one-dimensional (non-invertible) map is concerned, one might interpret this procedure as restoring the symmetry under time reversal. In instances where this procedure can be explicitly carried out at the level of the map, one obtains a new time-reversal symmetric triangular map on the unit square, which reduces to the original one-dimensional map after projecting along the appropriate dimension.

To provide an example, it is well-known that the angle-doubling Bernoulli map is equivalent to a coin-tossing game and that the equiprobability of all sequences of heads and tails amounts to the invariance of the Lebesgue measure on the interval. Extending the coin-tossing to memory-keeping doubly-infinite sequences, one realizes that this construction amounts to associating the baker map to the angle-doubling map, which is defined on the square and is time-reversal invariant. See for instance [5]. Notice that the invariant density is uniform, both with the one-dimensional and two-dimensional maps. One might pedantically say that the invariant density of the two-dimensional map is the product of that of the one-dimensional map evaluated along the two dimensions of the triangular map. Though this observation is trivial, it is indeed a case of the triangular map having an invariant density with what will be referred to as the product structure. This notion will play a central role in our discussion.

It is our purpose to show how this construction from one- to two-dimensional maps can be generalized and what properties of the invariant state can be inferred. Under specific symmetry assumptions on the map of the interval, we associate to it a two-dimensional triangular map of the unit square, symmetric under time-reversal. Considering the statistical properties of these maps, we establish the conditions so that the invariant measure of the triangular map has a smooth invariant density and identify the necessary and sufficient conditions under which this invariant density has a product structure in the sense defined above: namely it can be written as the product of the

density associated to the invariant measure of the one-dimensional map, evaluated along both dimensions.

As we describe below, the case where the invariant measure of the triangular map has a product form is arguably less interesting than when it does not. The result indeed suggests that, unless the measure has a product form, a complete statistical study of the expanding map of the interval requires considering its time-reversible triangular extension to the unit square. In other words there is more to learn about the statistical properties of one-dimensional map by studying the statistics of the associated two-dimensional map.

For the sake of illustration, we will consider in some detail a one-parameter class of time-reversal symmetric triangular map with a cusp. The interesting peculiarity of this class is that its limit upon variation of the parameter becomes intermittent. Yet all the maps of the class have equally uniform invariant densities on the unit interval [8]. This seemingly paradoxical property can be further explained provided one considers the corresponding two dimensional triangular map. As it turns out, the invariant measure develops a singularity as the parameter value tends to the intermittent limit.

The paper is organized as follows. The class of time-reversal symmetric triangular maps we will consider is defined in section 1. In section 2, we prove that the invariant measure of these maps is absolutely continuous. In section 3, we consider the special class of time-reversal symmetric triangular maps which are diffeomorphically conjugated to maps that preserve the volume measure and show that this condition is necessary and sufficient for the invariant measure to have the product form. Section 4 is devoted to a one-dimensional parameter class of time-reversal triangular maps which are generalizations of the one-dimensional cusp maps and establish key properties of their equilibrium states.

1. Time-reversible triangular maps of the square

Triangular or skew-product maps of the square are maps $F : [0,1]^2 \mapsto [0,1]^2$ of the form $F(x,y) = (f(x), g(x,y))$. A familiar such example is the baker map, which expands the square horizontally by a factor of 2 and squeezes it vertically by 1/2 so as to preserve areas, and subsequently folds the two horizontal halves on top of one another, thus recovering the unit square. Specifically, it is defined according to

$$B \;:\; (x,y) \mapsto \begin{cases} (2x, \frac{y}{2}), & 0 \leq x < 1/2, \\ (2x-1, \frac{y+1}{2}), & 1/2 \leq x < 1. \end{cases} \tag{1}$$

An important property of the baker map is that it is time-reversal symmetric, *i. e.* there exists an involution of the unit square, $T : [0,1]^2 \mapsto [0,1]^2$ such that

$$T \circ B \circ T = B^{-1}. \tag{2}$$

The baker map has actually two such symmetries, which map the square along its respective diagonals, $T_1(x,y) = (y,x)$ and $T_2(x,y) = (1-y, 1-x)$.

It is the purpose of this paper to establish the statistical properties of triangular maps of the square which are time reversal symmetric and can be obtained from the baker map whether through conjugation or continuous deformation.

More specifically, we consider maps of the form

$$F \; : \; (x, y) \mapsto \begin{cases} (f_0(x), g_0(y)), & 0 \le x < 1/2, \\ (f_1(x), g_1(y)), & 1/2 \le x < 1, \end{cases} \tag{3}$$

where, on the one hand, $f_0(x)$ is twice differentiable, strictly expanding, *i. e.* $f_0'(x) \ge \alpha > 1$, with $f_0(0) = 0$, $f_0(1/2) = 1$, and

$$f_1(x) = 1 - f_0(1 - x). \tag{4}$$

The y component of F, on the other hand, is defined through the inverse maps

$$\begin{aligned} g_0(x) &= f_0^{-1}(x), \\ g_1(x) &= f_1^{-1}(x). \end{aligned} \tag{5}$$

By construction, maps (3) have the hyperbolic properties and time-reversal symmetries of the baker map.

2. Absolutely continuous measure

Asymptotic statistical properties of maps (3) are determined by absolutely continuous measures whose densities $\rho(x, y)$ are invariant under the Perron Frobenius operator

$$\rho(x, y) = g_\omega'(x) f_\omega'(y) \rho(g_\omega(x), f_\omega(y)), \tag{6}$$

where $\omega = 0$ if $0 \le y < 1/2$ and $\omega = 1$ if $1/2 \le y < 1$. This is the content of our first

Theorem 1 *Let F be a triangular map of the square of the form (3), as specified above. The natural invariant measure of F is unique and absolutely continuous with respect to the volume measure, with density $\rho(x, y)$ whose marginals are equal,*

$$\int_0^1 \mathrm{d}y \; \rho(x, y) = \int_0^1 \mathrm{d}y \; \rho(y, x) \equiv \zeta(x), \tag{7}$$

where $\zeta(x)$ is the invariant density of the the one-dimensional map of the interval

$$f(x) = \begin{cases} f_0(x), & 0 \le x < 1/2, \\ f_1(x), & 1/2 \le x < 1. \end{cases} \tag{8}$$

To prove this result, we first notice that f as defined by (8) is an expanding piecewise C^2 map of the interval, so that, by the theorem of Lasota and Yorke [6], there is a unique L^1 integrable fixed point of the Perron-Frobenius operator, which we denote $\zeta(x)$,

$$\zeta(x) = g_0'(x)\zeta(g_0(x)) + g_1'(x)\zeta(g_1(x)). \tag{9}$$

This $\zeta(x)$ is the density associated to the natural invariant measure of f.

By extension, and since F is triangular with f its projection along the unstable direction, F has a unique Sinai-Ruelle-Bowen measure whose conditional measure along the unstable direction has density ζ, see [3] for a general discussion. Thus let ρ denote

the density associated to the SRB measure of F. The relation $\zeta(x) = \int_0^1 \mathrm{d}y \ \rho(x, y)$ ensues.

The second equality of equation (7), namely $\zeta(y) = \int_0^1 \mathrm{d}x \ \rho(x, y)$, is less immediate and can be derived starting with (6) evaluated at $(x, g_\omega(y))$. Integrating over x, we obtain

$$\begin{aligned} \int_0^1 \mathrm{d}x \ \rho(x, g_0(y)) &= f_0'(g_0(y)) \int_0^{1/2} \mathrm{d}x \ \rho(x, y), \\ \int_0^1 \mathrm{d}x \ \rho(x, g_1(y)) &= f_1'(g_1(y)) \int_{1/2}^1 \mathrm{d}x \ \rho(x, y). \end{aligned} \tag{10}$$

Using $f_\omega'(g_\omega(y)) = 1/g_\omega'(y)$, we can combine these two equations and infer the relation

$$\int_0^1 \mathrm{d}x \ \rho(x, y) = g_0'(y) \int_0^1 \mathrm{d}x \ \rho(x, g_0(y)) + g_1'(y) \int_0^1 \mathrm{d}x \ \rho(x, g_1(y)), \tag{11}$$

which completes the proof of equation (7).

Its marginals being identical, the invariant density therefore has the symmetries of F,

$$\rho(x, y) = \rho(y, x) = \rho(1 - y, 1 - x) = \rho(1 - x, 1 - y). \tag{12}$$

To finish the proof that $\rho(x, y)$ is absolutely continuous, it is sufficient to show that the phase-space contraction rate of F, or sum of the Lyapunov exponents, vanishes in average. Let λ_+ denote the positive Lyapunov exponent of F, equal to that of f above :

$$\lambda_+ = \int_0^{1/2} \mathrm{d}x \ \zeta(x) \log f_0'(x) + \int_{1/2}^1 \mathrm{d}x \ \zeta(x) \log f_1'(x) \tag{13}$$

The negative Lyapunov exponent is

$$\lambda_- = \int_0^{1/2} \mathrm{d}x \int_0^1 \mathrm{d}y \ \rho(x, y) \log g_0'(y) + \int_{1/2}^1 \mathrm{d}x \int_0^1 \mathrm{d}y \ \rho(x, y) \log g_1'(y), \tag{14}$$

From equation (10) above, $\int_0^{1/2} \mathrm{d}x \ \rho(x, y) = g_0'(y)\zeta(g_0(y))$ and $\int_{1/2}^1 \mathrm{d}x \ \rho(x, y) = g_1'(y)\zeta(g_1(y))$. Therefore

$$\begin{aligned} \lambda_- &= \int_0^1 \mathrm{d}x \ \Big[g_0'(x)\zeta(g_0(x)) \log g_0'(x) + g_1'(x)\zeta(g_1(x)) \log g_1'(x) \Big], \\ &= \int_0^{1/2} \mathrm{d}x \ \zeta(x) \log g_0'(f_0(x)) + \int_{1/2}^1 \mathrm{d}x \ \zeta(x) \log g_1'(f_1(x)), \\ &= -\int_0^{1/2} \mathrm{d}x \ \zeta(x) \log f_0'(x) - \int_{1/2}^1 \mathrm{d}x \ \zeta(x) \log f_1'(x), \\ &= -\lambda_+, \end{aligned} \tag{15}$$

where, in the third line, we used the identity $g_\omega'(f_\omega(x)) = 1/f_\omega'(x)$.

■

Triangular maps constructed upon one-dimensional differentiable expanding maps of the circle which are symmetric about 1/2 therefore have absolutely continuous invariant measures.

Notice that, upon inspection of equation (6), one might be led to believe that $\rho(x,y)$ has the product structure,

$$\rho(x,y) = \zeta(x)\zeta(y). \tag{16}$$

Indeed, since f and g are the inverses of one another, one may substitute $g_\omega(z)$ ($\omega = 0,1$, $0 \le z < 1$) for y in equation (6) and write

$$g'_\omega(z)\rho(x, g_\omega(z)) = g'_\omega(x)\rho(g_\omega(x), z). \tag{17}$$

This equation is symmetric between x and z, except for the value of ω, which is determined according to that of y:

$$\begin{aligned} \omega &= 0, \quad 0 \le y < 1/2, \\ \omega &= 1, \quad 1/2 \le y < 1. \end{aligned} \tag{18}$$

Therefore the product form (16) is in general invalid, unless equation (17) is independent of z, which requires the identity

$$g'_0(z)\zeta(g_0(z)) = g'_1(z)\zeta(g_1(z)). \tag{19}$$

As we will demonstrate shortly it is easy to find maps F of the form (3) which do not verify this property and consequently do not have a product measure.

The question which we address in the next section is to determine under which conditions the density $\rho(x,y)$ has the product form (16).

3. Diffeomorphic conjugations

An example of a non-trivial two-dimensional map for which equation (19) holds is that of the anti-symmetric logistic map. Let $f_0(x) = 4x(1-x)$ and f_1 defined as in equation (37). The inverses are $g_0(x) = 1/2(1-\sqrt{1-x})$ and $g_1(x) = 1/2(1+\sqrt{x})$ respectively. The invariant density of the one-dimensional map (8) is $\zeta(x) = 1/[\pi\sqrt{x(1-x)}]$ and one easily checks that $\rho(x,y) = \zeta(x)\zeta(y)$ verifies equation (6).

This property can be understood as a consequence of the diffeomorphic conjugation of the map above to the baker map. As is well-known [7], the one-dimensional logistic map above can be obtained from the angle doubling map by a conjugation,

$$f(x) = \phi^{-1}(2\phi(x) \bmod 1), \tag{20}$$

where $\phi(x) = 1 - 2/\pi \arccos\sqrt{x}$, with inverse $\phi^{-1}(x) = \cos^2[\pi(1-x)/2]$, and such that $\phi'(x) = \zeta(x)$.

We now set on to establish the generality of this result. Thus consider F as defined in (3) and let ϕ be a diffeomorphism of the unit interval with positive derivative. Set $\phi'(x) \equiv \sigma(x)$; σ is a positive density. Consider now the function

$$\widehat{F}(x,y) = \begin{cases} \Big(\phi\circ f_0\circ\phi^{-1}(x), \phi\circ g_0\circ\phi^{-1}(x)\Big), & 0 \le x < 1/2, \\ \Big(\phi\circ f_1\circ\phi^{-1}(x), \phi\circ g_1\circ\phi^{-1}(x)\Big), & 1/2 \le x < 1. \end{cases} \tag{21}$$

We further set $\widehat{f}_\omega \equiv \phi\circ f_\omega\circ\phi^{-1}$ and $\widehat{g}_\omega \equiv \phi\circ g_\omega\circ\phi^{-1}$, $\omega = 0,1$.

Lemma 2 *Let $\mathcal{P}_F$ and $\mathcal{P}_{\widehat{F}}$ denote the Perron Frobenius operators corresponding to F and $\widehat{F}$ respectively. Then*

$$\mathcal{P}_F\eta(x,y) = \mathcal{P}_\phi^{-1}\Big[\mathcal{P}_{\widehat{F}}\Big(\mathcal{P}_\phi\eta\Big)(x,y)\Big], \tag{22}$$

where

$$\mathcal{P}_\phi\eta(x,y) = \frac{\eta\Big(\phi^{-1}(x),\phi^{-1}(y)\Big)}{\sigma\Big(\phi^{-1}(x)\Big)\sigma\Big(\phi^{-1}(y)\Big)} \tag{23}$$

We have

$$\int_0^x \mathrm{d}s \int_0^y \mathrm{d}t\ \mathcal{P}_F\eta(s,t) = \int\int_{F^{-1}\left([0,x]\times[0,y]\right)} \mathrm{d}s\ \mathrm{d}t\ \eta(s,t), \tag{24}$$

where

$$F^{-1}([0,x]\times[0,y]) = \begin{cases} [0,g_0(x)]\times[0,f_0(y)], & 0\le y<1/2,\\ [0,g_0(x)]\times[0,1] & \\ \quad\cup[1/2,g_1(x)]\times[0,f_1(y)], & 1/2\le y<1. \end{cases} \tag{25}$$

Assuming $0\le y<1/2$, we can write

$$\begin{aligned}\int_0^x \mathrm{d}s \int_0^y \mathrm{d}t\ \mathcal{P}_F\eta(s,t) &= \int_0^{g_0(x)} \mathrm{d}s \int_0^{f_0(y)} \mathrm{d}t\ \eta(s,t),\\ &= \int_0^{\phi^{-1}\circ\widehat{g}_0\circ\phi(x)} \mathrm{d}s \int_0^{\phi^{-1}\circ\widehat{f}_0\circ\phi(y)} \mathrm{d}t\ \eta(s,t). \end{aligned}\tag{26}$$

Setting $s=\phi^{-1}(u)$ and $t=\phi^{-1}(v)$, the Jacobian of the transformation is $J=\phi'(s)\phi'(t)=\sigma(s)\sigma(t)$, so that the last integral can be rewritten

$$\begin{aligned}\int_0^{\widehat{g}_0\circ\phi(x)} \mathrm{d}u \int_0^{\widehat{f}_0\circ\phi(y)} \mathrm{d}v\ \frac{\eta(\phi^{-1}(u),\phi^{-1}(v))}{\sigma(\phi^{-1}(u))\sigma(\phi^{-1}(v))} &= \int_0^{\widehat{g}_0\circ\phi(x)} \mathrm{d}u \int_0^{\widehat{f}_0\circ\phi(y)} \mathrm{d}v\ \mathcal{P}_\phi\eta(u,v),\\ &= \int_0^{\phi(x)} \mathrm{d}u \int_0^{\phi(y)} \mathrm{d}v\ \mathcal{P}_{\widehat{F}}(\mathcal{P}_\phi\eta)(u,v),\\ &= \int_0^x \mathrm{d}s \int_0^y \mathrm{d}t\ \mathcal{P}_\phi^{-1}[\mathcal{P}_{\widehat{F}}(\mathcal{P}_\phi\eta)(s,t)]. \end{aligned}\tag{27}$$

This result holds for every $\eta\in L^1$. We therefore have derived the relation

$$\mathcal{P}_F\eta(x,y) = \mathcal{P}_\phi^{-1}[\mathcal{P}_{\widehat{F}}(\mathcal{P}_\phi\eta)(x,y)]. \tag{28}$$

The same relation holds for $1/2\le y<1$. ■

Theorem 3 *The invariant density $\rho(x,y)$ of F has the product form*

$$\rho(x,y) = \sigma(x)\sigma(y) = \phi'(x)\phi'(y). \tag{29}$$

if and only if $\widehat{F}$, which is obtained from the conjugation of F and ϕ, is measure-preserving, i. e. *$\widehat{F}$ has uniform invariant density, $\mathcal{P}_{\widehat{F}}1=1$.*

We have

$$\mathcal{P}_\phi(\sigma(x)\sigma(y)) = \frac{\sigma(\phi^{-1}(x))\sigma(\phi^{-1}(y))}{\sigma(\phi^{-1}(x))\sigma(\phi^{-1}(y))} = 1. \tag{30}$$

Thus, by the previous lemma,

$$\begin{aligned}\mathcal{P}_F(\sigma(x)\sigma(y)) &= \mathcal{P}_\phi^{-1}[\mathcal{P}_{\widehat{F}}(\mathcal{P}_\phi\sigma(x)\sigma(y))],\\ &= \mathcal{P}_\phi^{-1}[\mathcal{P}_{\widehat{F}}1],\\ &= \mathcal{P}_\phi^{-1}1,\\ &= \sigma(x)\sigma(y).\end{aligned} \tag{31}$$

The converse is true. Assume that F has invariant density $\rho(x, y) = \zeta(x)\zeta(y)$ and consider the homeomorphism

$$\phi(x) = \int_0^x \mathrm{d}s\ \zeta(s). \tag{32}$$

Then

$$\mathcal{P}_\phi\rho(x, y) = \frac{\zeta(\phi^{-1}(x))\zeta(\phi^{-1}(y))}{\zeta(\phi^{-1}(x))\zeta(\phi^{-1}(y))} = 1. \tag{33}$$

Let $\widehat{F}$ be the function constructed from F and the homeomorphism ϕ. We then have

$$\begin{aligned}\zeta(x)\zeta(y) &= \mathcal{P}_F(\zeta(x)\zeta(y)),\\ &= \mathcal{P}_\phi^{-1}[\mathcal{P}_{\widehat{F}}1].\end{aligned} \tag{34}$$

Assume that $\mathcal{P}_{\widehat{F}}1 \equiv d(x, y) \neq 1$. Then

$$\zeta(x)\zeta(y) = d(\phi(x), \phi(y))\zeta(x)\zeta(y), \tag{35}$$

a contradiction. ■

The invariant measures of maps that are conjugated to piecewise linear maps such as the baker map have therefore a product structure. And, conversely, a map whose invariant measure has a product structure is conjugated to a piecewise linear map by a diffeomorphism whose derivative is equal to the marginals of its probability density.

In the next section, we turn to a one-parameter family of maps (3) that does not have this property. They are obtained by continuous deformation of baker maps and, as the parameter is varied, have a singular limit which displays intermittency. As it will turn out, the marginals are uniform for all values of the parameter, yet the measure develops a singularity as one approaches the intermittent regime.

4. Cusp maps

We consider a two-dimensional extension of a class of anti-symmetric cusp maps, whose symmetric version was previously introduced in [8]. Let $0 < a \leq 1$. We define

$$f_0^{(a)}(x) = \frac{a+1}{2a}\left[1 - \sqrt{1 - \frac{8ax}{(a+1)^2}}\right], \tag{36}$$

for $0 \leq x \leq 1/2$, and

$$f_1^{(a)}(x) = 1 - f_0^{(a)}(1-x), \tag{37}$$

for $1/2 \leq x \leq 1$. The class of anti-symmetric cusp maps of the interval is defined by

$$f^{(a)} : \ x \mapsto \begin{cases} f_0^{(a)}(x), & 0 \leq x \leq 1/2, \\ f_1^{(a)}(x), & 1/2 < x \leq 1. \end{cases} \tag{38}$$

Figure 1 displays several of these functions as the parameter a is varied between 0 and 1.

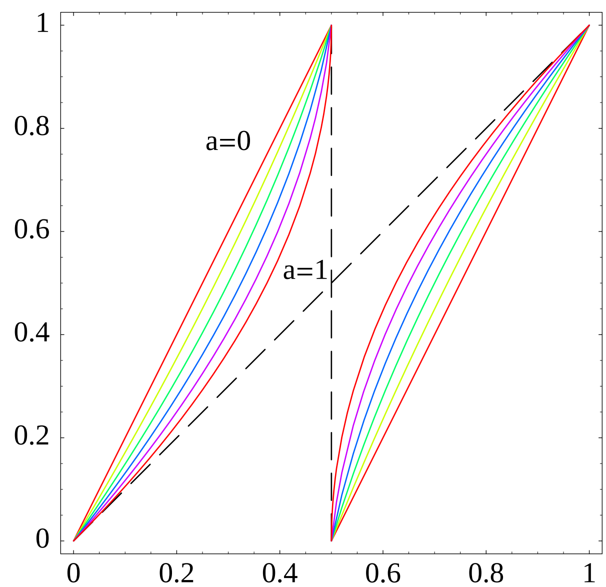

Figure 1. Anti-symmetric cusps maps (38), here shown for $a = 0, 1/5, \ldots, 1$.

The inverses of $f_0^{(a)}$ and $f_1^{(a)}$ are

$$\begin{aligned} g_0^{(a)}(x) &\equiv f_0^{(a)\,-1} = \frac{1+a}{2}x - \frac{a}{2}x^2, \\ g_1^{(a)}(x) &\equiv f_1^{(a)\,-1} = \frac{1}{2} + \frac{1-a}{2}x + \frac{a}{2}x^2. \end{aligned} \tag{39}$$

An immediate property of the Perron-Frobenius operators attached to the maps (38) is that they preserve the Lebesgue measure and therefore have uniform density, irrespective of the value of a. This follows from equation (9) and the identity

$$g_0^{(a)\prime}(x) + g_1^{(a)\prime}(x) = \frac{a+1}{2} - ax + ax + \frac{1-a}{2} = 1, \tag{40}$$

where the prime indicates the derivative with respect to the argument. Thus the computation of the positive Lyapunov exponent is straightforward and yields

$$\lambda_+^{(a)} = \log 2 + \frac{1}{2} + \frac{(1-a)^2}{4a}\log(1-a) - \frac{(1+a)^2}{4a}\log(1+a). \tag{41}$$

In particular $\lambda_+^{(0)} = \log 2$ and $\lambda_+^{(1)} = 1/2$.

The specificity of the maps $f^{(a)}$ thus defined is that they can be viewed, as one tunes the value of the parameter a from 0 to 1, as continuous deformations of the angle-doubling map, $x \mapsto 2x$ if $0 \leq x < 1/2$, or $2x-1$ if $1/2 \leq x < 1$, to the intermittent anti-symmetric cusp map, $x \mapsto 1-\sqrt{1-2x}$ or $\sqrt{2x-1}$, depending on whether $x < 1/2$ or $x \geq 1/2$. The latter case is weakly intermittent in the sense that the slope of $f^{(1)}$ at $x = 0$ is unity, and yet the invariant density is constant and therefore shows no sign of the singularity that underlies the intermittency of its statistical observables. Nevertheless this regime is characterized by the power law decay of correlation functions, which stems from the existence of an accumulation of the eigenvalue spectrum of the Frobenius-Perron operator towards the eigenvalue 1, which corresponds to the stationary state. See refs. [9, 10, 11, 12, 13, 14] for the treatment of the symmetric case.

In order to display the effect of the arising intermittency as $a \to 1$ on the statistical properties of trajectories driven by the maps (38) and the singularity of the invariant measure in the intermittent regime ($a = 1$), one needs to consider the two-dimensional extension of $f^{(a)}$ to time-reversible triangular maps, defined in accordance to equation (3) :

$$F^{(a)}: \ (x,\, y) \mapsto \begin{cases} \left(f_0^{(a)}(x),\ g_0^{(a)}(y)\right), & 0 \leq x \leq 1/2, \\ \left(f_1^{(a)}(x),\ g_1^{(a)}(y)\right), & 1/2 < x \leq 1. \end{cases} \tag{42}$$

By construction, $F^{(a)}$, except at $a = 1$, has the properties studied in section 2 and, in particular, verifies theorem 1, asserting that the invariant density is smooth, with both marginals trivial. In particular $\lambda_-^{(a)} = -\lambda_+^{(a)}$.

From equation (6), the invariant density verifies the functional equation

$$\rho_a(x,y) = \begin{cases} \frac{1+a-2ax}{\sqrt{(1+a)^2-8ay}}\rho_a\left(\frac{1+a}{2}x - \frac{a}{2}x^2,\, \frac{1+a-\sqrt{(1+a)^2-8ay}}{2a}\right), & 0 \leq y \leq 1/2, \\ \frac{1-a+2ax}{\sqrt{(1+a)^2-8a(1-y)}}\rho_a\left(\frac{1}{2} + \frac{1-a}{2}x + \frac{a}{2}x^2,\, \frac{\sqrt{(1+a)^2-8a(1-y)}-1+a}{2a}\right), & 1/2 \leq y \leq 1. \end{cases} \tag{43}$$

A remarkable property of ρ_a, that can be seen in figure 2, is that it develops a singularity at the origin as $a \to 1$.

In order to prove this assertion, we consider the partial cumulative function

$$r_a(x,y) \equiv \int_0^y \mathrm{d}y\ \rho_a(x,y). \tag{44}$$

From equation (43), we obtain the following functional equation for this quantity :

$$r_a(x,y) = \begin{cases} \left(\frac{1+a}{2} - ax\right) r_a\left(\frac{1+a}{2}x - \frac{a}{2}x^2,\, \frac{a+1-\sqrt{(1+a)^2-8ay}}{2a}\right), \\ \qquad\qquad 0 \leq y \leq 1/2, \\ \frac{1+a}{2} - ax + \left(\frac{1-a}{2} + ax\right) r_a\left(\frac{1}{2} + \frac{1-a}{2}x + \frac{a}{2}x^2,\, \frac{\sqrt{(1+a)^2-8a(1-y)}-1+a}{2a}\right), \\ \qquad\qquad 1/2 \leq y \leq 1. \end{cases} \tag{45}$$

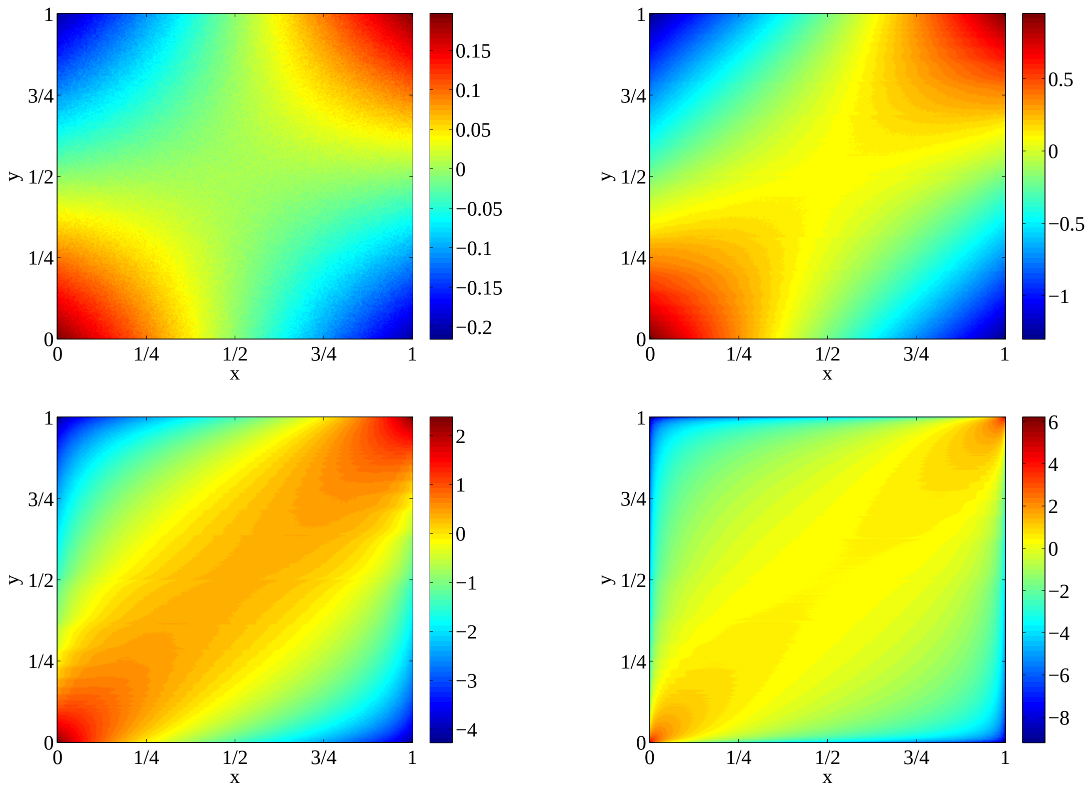

Figure 2. Numerical computations of the invariant density $\rho_a(x, y)$, where a takes the values $a = 0.1$, 0.5, 0.9, and 1, from left to right and top to bottom. This histogram is computed from time series of many trajectories with initial conditions uniformly distributed over the square. The grid size is 300×300 cells, except for the last one which uses a grid of 1000×1000 cells. The colour code uses a logarithmic scale, different for each plot, *i. e.* the legends refer to the natural logarithms of the density values. The lower left and upper right corners correspond to higher densities, the lower right and upper left corners to lower densities.

By construction, we have the properties

$$r_a(x, 1) = 1, \tag{46}$$

$$\int_0^1 \mathrm{d}x \; r_a(x, y) = y. \tag{47}$$

Numerical computations of these quantities are displayed in figure 3.

The functional Equation (45) has a form suited to the computation of the invariant state. Indeed, using equation (46), we can infer the expressions of $r_a(x, y)$ at the points y which are pre-images of $y = 1$ with respect to $f_\omega^{(a)}$, $\omega = 0, 1$. In particular, it is immediate to see that $r_a(x, 1/2) = {g_0^{(a)}}'(x) = (1 + a)/2 - ax$. In general, the pre-images of $y = 1$ are the points $g_{\underline{\omega}_n}^{(a)}(1) \equiv g_{\omega_n}^{(a)} \circ \ldots \circ g_{\omega_1}^{(a)}(1)$, where $\underline{\omega}_n$ is a compact notation for the sequence of n digits ω_i, $\omega_i = 0$ or 1 according to which branch $g_0^{(a)}$ or $g_1^{(a)}$ is to be iterated.

Lemma 4 *Fix $0 < a < 1$. Given any y_0 in the unit interval and $\varepsilon \ll 1$, one can find an integer n and a sequence $\underline{\omega}_n$ such that $g_{\underline{\omega}_n}^{(a)}(1) \in (y_0 - \varepsilon, y_0 + \varepsilon)$.*

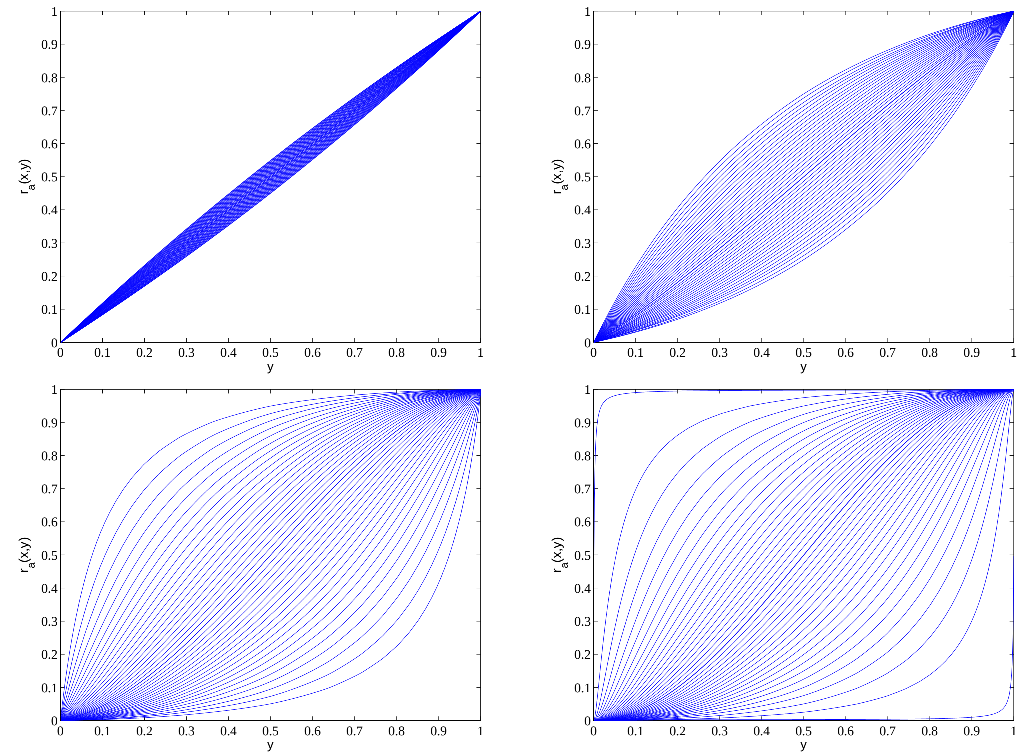

Figure 3. Cumulative functions $r_a(x, y)$ vs. y, with $a = 0.1$, 0.5, 0.9 and 1, from top to bottom and left to right. The different curves correspond to different values of x uniformly spread over the interval, excluding $x = 0$ and $x = 1$. The curves are ordered with growing values of x from right to left.

The proof is a consequence of the existence of Markov partitions for the maps (38), $a < 1$, and relies on the contraction of the inverses, ${g_0^{(a)}}'(x) < 1$ and ${g_1^{(a)}}'(x) < 1$. ■

This implies that $r_a(x, y)$, for fixed x and $a < 1$, is completely determined by equation (45).

We are specifically interested in those pre-images of $y = 1$, which lay closest to the origin, *i. e.* $y = g_0^{(a)} \circ \ldots \circ g_0^{(a)}(1)$. Thus considering the left branch of equation (45), we have

$$\begin{aligned}
r_a\Big(x, g_{\underline{0}_n}^{(a)}(1)\Big) &= {g_0^{(a)}}'(x) r_a\Big(g_0^{(a)}(x), g_{\underline{0}_{n-1}}^{(a)}(1)\Big), \\
&= \prod_{k=0}^{n-1} {g_0^{(a)}}'\Big(g_{\underline{0}_k}^{(a)}(x)\Big), \\
&= r_a\Big(x, g_{\underline{0}_{n-1}}^{(a)}(1)\Big) {g_0^{(a)}}'\Big(g_{\underline{0}_{n-1}}^{(a)}(x)\Big), \\
&= r_a\Big(x, g_{\underline{0}_{n-1}}^{(a)}(1)\Big) \left(\frac{a+1}{2} - a g_{\underline{0}_{n-1}}^{(a)}(x)\right).
\end{aligned} \tag{48}$$

These steps easily generalize to any symbolic sequence $\underline{\omega}_n$ for which we can write

$$r_a\Big(x, g_{\underline{\omega}_n}^{(a)}(1)\Big) = {g_{\omega_n}^{(a)}}'(x) r_a\Big(g_{\omega_n}^{(a)}(x), g_{\underline{\omega}_{n-1}}^{(a)}(1)\Big) + \omega_n {g_{1-\omega_n}^{(a)}}'(x). \tag{49}$$

Thus, fixing x and starting at $y = 1$, we can use the above equations to compute points on the curves displayed on the right panels of figure 3.

Setting $x = 0$ in equation (49), we have

$$r_a\Big(0, g^{(a)}_{\underline{\omega}_n}(1)\Big) = g^{(a)}_{\omega_n}{}'(0) r_a\Big(g^{(a)}_{\omega_n}(0), g^{(a)}_{\underline{\omega}_{n-1}}(1)\Big) + \omega_n g^{(a)}_{1-\omega_n}{}'(0). \tag{50}$$

Thus

$$r_a\Big(0, g^{(a)}_{\underline{\omega}_n}(1)\Big) = \begin{cases} \frac{a+1}{2} r_a\Big(0, g^{(a)}_{\underline{\omega}_{n-1}}(1)\Big), & \omega_n = 0, \\ \frac{a-1}{2} r_a\Big(\frac{1}{2}, g^{(a)}_{\underline{\omega}_{n-1}}(1)\Big) + \frac{a+1}{2}, & \omega_n = 1. \end{cases} \tag{51}$$

Taking the limit as $a \to 1$, we have

$$r_1\Big(0, g^{(1)}_{\underline{\omega}_n}(1)\Big) = \begin{cases} r_1\Big(0, g^{(1)}_{\underline{\omega}_{n-1}}(1)\Big), & \omega_n = 0, \\ 1, & \omega_n = 1. \end{cases} \tag{52}$$

Both these two alternatives yield

$$r_1\Big(0, g^{(1)}_{\underline{\omega}_n}(1)\Big) = 1. \tag{53}$$

Indeed, if one amongst the $\omega_{n-1}, \dots, \omega_1$ is equal to 1, the first alternative eventually reduces to the second; if on the other hand $\underline{\omega}_n = \underline{0}_n$, we get

$$r_1\Big(0, g^{(1)}_{\underline{0}_n}(1)\Big) = r_1\Big(0, g^{(1)}_{0}(1)\Big) = 1. \tag{54}$$

Thus

$$\lim_{a\to 1} r_a(0, y) = 1, \tag{55}$$

and, in particular $r_1(0, 0) = 1$. Therefore, in the intermittent regime, $a = 1$, the invariant density has a singularity at the origin,

$$r_1(x, 0) = \begin{cases} 1, & x = 0, \\ 0, & x > 0, \end{cases} \tag{56}$$

which is otherwise absent in the hyperbolic regime, $a < 1$, for which we have $r_a(x, 0) = 0$, $x = 0$ included.

The latter property can easily be checked using simple arguments. Indeed the density at the origin is

$$\begin{aligned} \rho_a(0,0) &= \lim_{n\to\infty} \frac{r_a\Big(0, g^{(a)}_{\underline{0}_{n+1}}(1)\Big) - r_a\Big(0, g^{(a)}_{\underline{0}_n}(1)\Big)}{g^{(a)}_{\underline{0}_{n+1}}(1) - g^{(a)}_{\underline{0}_n}(1)}, \\ &= \lim_{n\to\infty} \frac{(1-a) r_a\Big(0, g^{(a)}_{\underline{0}_n}(1)\Big)}{(1-a) g^{(a)}_{\underline{0}_n}(1) + a\Big[g^{(a)}_{\underline{0}_n}(1)\Big]^2}, \\ &= \lim_{n\to\infty} \frac{r_a\Big(0, g^{(a)}_{\underline{0}_n}(1)\Big)}{g^{(a)}_{\underline{0}_n}(1)}, \\ &= \lim_{n\to\infty} \frac{[(1+a)/2]^n}{g^{(a)}_{\underline{0}_n}(1)}. \end{aligned} \tag{57}$$

which, as proven in [15, Theorem 2.1], exists and is finite for $a < 1$. Furthermore, we expect, though we have no formal proof of this result at this point, that, as $a \to 1$, $\rho_a(0,0)$ diverges as $1/(1-a)$.

5. Conclusions

In this paper, we considered the smooth invariant statistics of time-reversal symmetric triangular maps of the unit square built upon anti-symmetric piecewise expanding maps of the unit interval.

We showed that maps which are diffeomorphically conjugated to piecewise linear maps have an equilibrium state with the product form, simply expressed as the product of the derivative of the conjugating map, evaluated at the two variables.

Maps whose invariant state has the product form are therefore exceptional. For piecewise expanding maps that are not diffeomorphically conjugated to piecewise linear maps, a thorough study of their statistical properties can only be properly accomplished provided one considers the map from the interval to the square thus recovering a time-reversal symmetric map.

The example of the class of anti-symmetric cusp maps considered in this paper is revealing in that respect. Though the natural invariant measures of the one-dimensional maps of this class have uniform densities, even in the intermittent regime, the equilibrium state of the associated time-reversible two-dimensional map displays a singularity at the intermittent fixed point.

In the non-equilibrium physics literature, time-reversible systems submitted to non-holonomic constraints have been considered in the context of non-equilibrium molecular dynamics. Methods were developed over the last decades using iso-kinetic thermostats under Gauss's principle of least constraint [16], or Nosé-Hoover thermostats [17]. The interesting point with regards to the results presented in this paper is that the equilibrium states of systems subjected to such non-holonomic constraints are not uniform. This happens because phase-space volumes are not preserved pointwise, though, in average, they are [18]. These equilibrium states therefore share the properties of the invariant measures of the maps considered in this paper. Our results suggest that these states will in general not be factorisable and display a rich structure.

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